



THE DIFFERENTIAL GEOMETRY OF PARAMETRIC PRIMITIVES

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Abstract: We derive the expressions for first and second derivatives, normal, metric matrix and curvature matrix for spheres, cones, cylinders, and tori.

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Differential Properties of Parametric Surfaces

A parametric surface is a function:

$$\mathbf{x} = \mathbf{F}(\mathbf{u})$$

where

$$\mathbf{x} = [x \ y \ z]$$

is a point in affine 3-space, and

$$\mathbf{u} = [u \ v]$$

is a point in affine 2-space.

The *Jacobian* matrix is a matrix of partial derivatives that relate changes in u and v to changes in x , y , and z :

$$\mathbf{J} = \frac{\partial(x, y, z)}{\partial(u, v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial u} \\ \frac{\partial \mathbf{x}}{\partial v} \end{bmatrix}$$

The Hessian is a tensor of second partial derivatives:

$$\mathbf{H} = \frac{\partial^2(x, y, z)}{\partial(u, v) \partial(u, v)} = \begin{bmatrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 y}{\partial u^2} & \frac{\partial^2 z}{\partial u^2} \\ \frac{\partial^2 x}{\partial u \partial v} & \frac{\partial^2 y}{\partial u \partial v} & \frac{\partial^2 z}{\partial u \partial v} \\ \frac{\partial^2 x}{\partial v \partial u} & \frac{\partial^2 y}{\partial v \partial u} & \frac{\partial^2 z}{\partial v \partial u} \\ \frac{\partial^2 x}{\partial v^2} & \frac{\partial^2 y}{\partial v^2} & \frac{\partial^2 z}{\partial v^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial^2 \mathbf{x}}{\partial u^2} & \frac{\partial^2 \mathbf{x}}{\partial u \partial v} \\ \frac{\partial^2 \mathbf{x}}{\partial v \partial u} & \frac{\partial^2 \mathbf{x}}{\partial v^2} \end{bmatrix}$$

The first fundamental form is defined as:

$$\mathbf{G} = \mathbf{J} \mathbf{J}^t = \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{x}}{\partial u} & \frac{\partial \mathbf{x}}{\partial u} \cdot \frac{\partial \mathbf{x}}{\partial v} \\ \frac{\partial \mathbf{x}}{\partial v} \cdot \frac{\partial \mathbf{x}}{\partial u} & \frac{\partial \mathbf{x}}{\partial v} \cdot \frac{\partial \mathbf{x}}{\partial v} \end{bmatrix}$$

and establishes a metric of differential length:

$$(dx)^2 = (du)G(du)^t$$

so that the arc length of a curve segment, $\mathbf{u} = \mathbf{u}(t)$, $t_0 < t < t_1$ is given by:

$$s = \int_{t_0}^{t_1} \frac{ds}{dt} dt = \int_{t_0}^{t_1} |\dot{\mathbf{x}}| dt = \int_{t_0}^{t_1} |\dot{\mathbf{x}}| dt = \int_{t_0}^{t_1} (\dot{\mathbf{u}}G\dot{\mathbf{u}})^{\frac{1}{2}} dt$$

The differential surface area enclosed by the differential parallelogram (du, dv) is approximately:

$$dS^a = (|G|)^{\frac{1}{2}} du dv$$

so that the area of a region of the surface corresponding to a region R in the u - v plane is:

$$S = \int_R (|G|)^{\frac{1}{2}} du dv$$

The second fundamental matrix measures normal curvature, and is given by:

$$D = \mathbf{n} \Sigma H = \begin{pmatrix} \mathbf{n} \Sigma \frac{\partial^2 \mathbf{x}}{\partial u^2} & \mathbf{n} \Sigma \frac{\partial^2 \mathbf{x}}{\partial u \partial v} \\ \mathbf{n} \Sigma \frac{\partial^2 \mathbf{x}}{\partial v \partial u} & \mathbf{n} \Sigma \frac{\partial^2 \mathbf{x}}{\partial v^2} \end{pmatrix}$$

The normal curvature is defined to be positive a curve $\mathbf{u}$ on the surface turns toward the positive direction of the surface normal by:

$$k_n = \frac{\dot{\mathbf{u}} D \dot{\mathbf{u}}^t}{\dot{\mathbf{u}} G \dot{\mathbf{u}}^t}$$

The deviation (in the normal direction) from the tangent plane of the surface, given a differential displacement of $\dot{\mathbf{u}}$ is:

$$\ddot{\mathbf{x}} \Sigma \mathbf{n} = \dot{\mathbf{u}} D \dot{\mathbf{u}}^t$$

Reparametrization

If the parametrization of the surface is transformed by the equations:

$$u' = u'(u, v) \quad \text{and} \quad v' = v'(u, v)$$

then the chain rule yields:

$$\frac{\partial(x, y, z)}{\partial(u', v')} = \frac{\partial(u, v)}{\partial(u', v')} \frac{\partial(x, y, z)}{\partial(u, v)}$$

or

$$\mathbf{J}' = \mathbf{P} \mathbf{J}$$

where

$$\mathbf{J}' = \frac{\partial(x, y, z)}{\partial(u', v')}$$

is the new Jacobian matrix of the surface with respect to the new parameters u' and v' , and

$$\mathbf{P} = \frac{\partial(u, v)}{\partial(u', v')} = \begin{bmatrix} \frac{\partial u}{\partial u'} & \frac{\partial v}{\partial u'} \\ \frac{\partial u}{\partial v'} & \frac{\partial v}{\partial v'} \end{bmatrix}$$

is the Jacobian matrix of the reparametrization.

The new Hessian is given by

$$\mathbf{H}' = \mathbf{P} \mathbf{H} \mathbf{P}^T + \mathbf{Q} \mathbf{J}$$

where

$$\mathbf{Q} = \begin{bmatrix} \frac{\partial^2(u, v)}{\partial u'^2} & \frac{\partial^2(u, v)}{\partial u' \partial v'} \\ \frac{\partial^2(u, v)}{\partial v' \partial u'} & \frac{\partial^2(u, v)}{\partial v'^2} \end{bmatrix}$$

The new fundamental matrix is given by:

$$\mathbf{G}' = \mathbf{P} \mathbf{G} \mathbf{P}^T$$

and the new curvature matrix is given by:

$$\mathbf{D}' = \mathbf{P} \mathbf{D} \mathbf{P}^T$$

Change of Coordinates

For simplicity, we have defined several primitives with unit size, located at the origin. Related to the reparametrization is the change of coordinates $\mathbf{x} \Leftarrow \mathbf{x}(\xi, \eta)$, with associated Jacobian:

$$\mathbf{C} = \frac{\partial \mathbf{x}}{\partial \xi} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}$$

When the change of coordinates is represented by the affine transformation:

$$\mathbf{A} = \begin{bmatrix} \tilde{x}_x & y_x & z_x \\ \tilde{x}_y & y_y & z_y \\ \tilde{x}_z & y_z & z_z \\ \tilde{x}_0 & y_0 & z_0 \end{bmatrix}$$

the Jacobian is simply the submatrix:

$$\mathbf{C} = \begin{bmatrix} \tilde{x}_x & y_x & z_x \\ \tilde{x}_y & y_y & z_y \\ \tilde{x}_z & y_z & z_z \end{bmatrix}$$

Regardless, the Jacobian and Hessian transform as follows:

$$\mathbf{J} \Leftarrow \mathbf{J}\mathbf{C}, \quad \mathbf{H} \Leftarrow \mathbf{H}\mathbf{C}$$

The normal is transformed as:

$$\mathbf{n} \Leftarrow \frac{\mathbf{n}\mathbf{C}^{-1t}}{(\mathbf{n}\mathbf{C}^{-1t}\mathbf{C}^{-1}\mathbf{n})^{\frac{1}{2}}}$$

The denominator arises from the desire to have a *unit* normal.

The first and second fundamental matrices are then calculated as:

$$\mathbf{G} \Leftarrow \mathbf{J}\mathbf{C}\mathbf{J}^t = \mathbf{J}\mathbf{C}\mathbf{C}^t\mathbf{J}^t$$

$$\mathbf{D} \Leftarrow \mathbf{H}\mathbf{C}\sum \mathbf{n} \Leftarrow \frac{(\mathbf{H}\mathbf{C})\sum(\mathbf{n}\mathbf{C}^{-1t})}{(\mathbf{n}\mathbf{C}^{-1t}\mathbf{C}^{-1}\mathbf{n})^{\frac{1}{2}}} = \frac{\mathbf{H}\mathbf{C}\mathbf{C}^{-1}\mathbf{n}^t}{(\mathbf{n}\mathbf{C}^{-1t}\mathbf{C}^{-1}\mathbf{n})^{\frac{1}{2}}} = \frac{\mathbf{H}\sum \mathbf{n}}{(\mathbf{n}\mathbf{C}^{-1t}\mathbf{C}^{-1}\mathbf{n})^{\frac{1}{2}}} = \frac{\mathbf{D}}{(\mathbf{n}\mathbf{C}^{-1t}\mathbf{C}^{-1}\mathbf{n})^{\frac{1}{2}}}$$

Not very pretty. But certain types of transformations can be applied easily. For a uniform scale with arbitrary translations,

$$\mathbf{C} = \begin{bmatrix} \tilde{x} & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} = r\mathbf{I}$$

so that

$$\mathbf{J}\phi = r\mathbf{J}, \quad \mathbf{H}\phi = r\mathbf{H}, \quad \mathbf{n}\phi = \mathbf{n}, \quad \mathbf{G}\phi = r^2\mathbf{G}, \quad \mathbf{D}\phi = r\mathbf{D}$$

For rotations (and arbitrary translations), the Jacobian matrix $\mathbf{C}=\mathbf{R}$ is orthogonal, so the inverse is equal to the transpose, yielding:

$$\mathbf{J}\phi = \mathbf{J}\mathbf{R}, \quad \mathbf{H}\phi = \mathbf{H}\mathbf{R}, \quad \mathbf{n}\phi = \mathbf{n}\mathbf{R}, \quad \mathbf{G}\phi = \mathbf{G}, \quad \mathbf{D}\phi = \mathbf{D}$$

Combining the two, we have the results for a transformation that includes translations, rotations and uniform scale:

$$\mathbf{J}\phi = r\mathbf{J}\mathbf{R}, \quad \mathbf{H}\phi = r\mathbf{H}\mathbf{R}, \quad \mathbf{n}\phi = \mathbf{n}\mathbf{R}, \quad \mathbf{G}\phi = r^2\mathbf{G}, \quad \mathbf{D}\phi = r\mathbf{D}$$

or in terms of the composite matrix $\mathbf{C} = r\mathbf{R}$:

$$\mathbf{J}\phi = \mathbf{J}\mathbf{C}, \quad \mathbf{H}\phi = \mathbf{H}\mathbf{C}, \quad \mathbf{n}\phi = \frac{\mathbf{n}\mathbf{C}}{(|\mathbf{C}|)^{\frac{1}{3}}}, \quad \mathbf{G}\phi = (|\mathbf{C}|)^{\frac{2}{3}}\mathbf{G}, \quad \mathbf{D}\phi = (|\mathbf{C}|)^{\frac{1}{3}}\mathbf{D}$$

Sphere

Given the spherical coordinates:

$$[x \ y \ z] = [r \sin \theta \cos \phi \ r \sin \theta \sin \phi \ r \cos \theta]$$

we have the Jacobian matrix:

$$\frac{\partial(x, y, z)}{\partial(\theta, \phi)} = \begin{bmatrix} -r \sin \theta & r \cos \theta \cos \phi & r \cos \theta \sin \phi \\ r \cos \theta \cos \phi & -r \sin \theta \cos \phi & -r \sin \theta \sin \phi \\ r \cos \theta \sin \phi & r \sin \theta \cos \phi & r \sin \theta \sin \phi \end{bmatrix}$$

the Hessian tensor:

$$\frac{\partial^2(x, y, z)}{\partial(\theta, \phi) \partial(\theta, \phi)} = \begin{bmatrix} -r & 0 & 0 \\ 0 & -r \sin \theta \cos \phi & -r \sin \theta \sin \phi \\ 0 & r \sin \theta \cos \phi & r \sin \theta \sin \phi \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G} = \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2 \theta \end{bmatrix}$$

the normal:

$$\mathbf{n} = \frac{1}{r} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and the second fundamental form:

$$\mathbf{D} = \begin{bmatrix} -\frac{x}{r} & -\frac{y}{r} & -\frac{z}{r} \\ -\frac{y}{r} & \frac{x}{r} & 0 \\ -\frac{z}{r} & 0 & \frac{y}{r} \end{bmatrix}$$

Unit Sphere

Angle Parametrization

Given the unit spherical coordinates with $0 \leq \varphi < 2\pi$, $0 \leq \theta < \pi$, we parametrize the sphere:

$$[x \ y \ z] = [\sin \theta \cos \varphi \ \sin \theta \sin \varphi \ \cos \theta]$$

This yields the Jacobian matrix:

$$\mathbf{J}_{\varphi\theta} = \begin{bmatrix} -\sin \theta \sin \varphi & \sin \theta \cos \varphi & 0 \\ \cos \theta \cos \varphi & \cos \theta \sin \varphi & -\sin \theta \\ \sin \varphi & \cos \varphi & 0 \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H}_{\varphi\theta} = \begin{bmatrix} -\sin \theta \cos \varphi & -\sin \theta \sin \varphi & 0 \\ \cos \theta \sin \varphi & \cos \theta \cos \varphi & -\sin \theta \\ \cos \varphi & -\sin \varphi & 0 \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G}_{\varphi\theta} = \begin{bmatrix} \sin^2 \theta & 0 \\ 0 & 1 \end{bmatrix}$$

the normal:

$$\mathbf{n} = [x \ y \ z]$$

and the second fundamental form:

$$\mathbf{D}_{\varphi\theta} = \begin{bmatrix} \sin \theta \cos \varphi & \sin \theta \sin \varphi \\ 0 & -\sin \theta \end{bmatrix}$$

Angle Parametrization

With the reparametrization $\varphi = 2\pi u$, $\theta = \pi v$, we have the Jacobian:

$$\mathbf{P} = \begin{bmatrix} 2\pi \sin \pi v \cos 2\pi u & 2\pi \sin \pi v \sin 2\pi u \\ 0 & -2\pi \cos \pi v \end{bmatrix}$$

Applying the chain rule, we have:

$$\mathbf{J}_{uv} = \begin{bmatrix} -2\pi \sin \pi v \sin 2\pi u & 2\pi \sin \pi v \cos 2\pi u & 0 \\ 2\pi \cos \pi v \cos 2\pi u & 2\pi \cos \pi v \sin 2\pi u & -2\pi \sin \pi v \\ 2\pi \cos 2\pi u & -2\pi \sin 2\pi u & 0 \end{bmatrix}$$

$$\mathbf{H}_{uv} = \begin{bmatrix} \ddot{E} & 4p^2[-x & -y & 0] \\ \vdots & 2p\hat{I} \begin{bmatrix} -\frac{yz}{\sqrt{x^2+y^2}} & -\frac{xz}{\sqrt{x^2+y^2}} & 0 \end{bmatrix} \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix}$$

$$\mathbf{G}_{uv} = \begin{bmatrix} \ddot{E} 4p^2(x^2 + y^2) & 0 \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix}$$

$$\mathbf{D}_{uv} = \begin{bmatrix} \ddot{E} 4p^2(x^2 + y^2) & 0 \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix}$$

Changing coordinates to yield a sphere of arbitrary radius, we find that the expressions for the Jacobian, the Hessian, and the metric matrix remain the same, because x , y , and z scale linearly with r . The curvature matrix changes to:

$$\mathbf{D}_{uv} = \begin{bmatrix} \ddot{E} \frac{4p^2(x^2 + y^2)}{r} & 0 \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix}$$

Cone

Angle Parametrization

Given the unit conical parametrization:

$$[x \ y \ z] = [z \cos q \ z \sin q \ z]$$

we have the Jacobian matrix:

$$\mathbf{J}_{qz} = \begin{bmatrix} \frac{\partial x}{\partial q} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial q} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial q} & \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} -y & x \\ z & z \\ 0 & 1 \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H}_{qz} = \begin{bmatrix} \frac{\partial^2 x}{\partial q^2} & \frac{\partial^2 x}{\partial q \partial z} \\ \frac{\partial^2 y}{\partial q^2} & \frac{\partial^2 y}{\partial q \partial z} \\ \frac{\partial^2 z}{\partial q^2} & \frac{\partial^2 z}{\partial q \partial z} \end{bmatrix} = \begin{bmatrix} -x & -y \\ \frac{y}{z} & \frac{x}{z} \\ 0 & 0 \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G}_{qz} = \begin{bmatrix} \frac{\partial x^2}{\partial q^2} + \frac{\partial y^2}{\partial q^2} & 0 \\ 0 & \frac{x^2 + y^2 + z^2}{z^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial z^2}{\partial q^2} & 0 \\ 0 & 2 \end{bmatrix}$$

the normal:

$$\mathbf{n}_{qz} = \begin{bmatrix} \frac{\partial x}{\partial q} \\ \frac{\partial y}{\partial q} \\ \frac{\partial z}{\partial q} \end{bmatrix} = \begin{bmatrix} -y \\ z \\ 0 \end{bmatrix}$$

and the second fundamental form:

$$\mathbf{D}_{qz} = \begin{bmatrix} \frac{\partial^2 x}{\partial q^2} \\ \frac{\partial^2 x}{\partial q \partial z} \\ \frac{\partial^2 y}{\partial q^2} \\ \frac{\partial^2 y}{\partial q \partial z} \\ \frac{\partial^2 z}{\partial q^2} \\ \frac{\partial^2 z}{\partial q \partial z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Unit Parametrization

For the parametrization:

$$[x \ y \ z] = [rv \cos 2pu \ rv \sin 2pu \ vh]$$

we have:

$$\mathbf{J}_{uv} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} = \begin{bmatrix} -2py & 2px \\ \frac{hx}{rz} & \frac{hy}{rz} \\ h & h \end{bmatrix}$$

$$\mathbf{H}_{uv} = \begin{bmatrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 x}{\partial u \partial v} \\ \frac{\partial^2 y}{\partial u^2} & \frac{\partial^2 y}{\partial u \partial v} \\ \frac{\partial^2 z}{\partial u^2} & \frac{\partial^2 z}{\partial u \partial v} \end{bmatrix} = \begin{bmatrix} -4p^2x & -4p^2y \\ \frac{2ph}{rz}[-y \ x \ 0] & \frac{2ph}{rz}[-y \ x \ 0] \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{G}_{uv} = \begin{pmatrix} 4p^2(x^2 + y^2) & 0 \\ 0 & \frac{h^2(x^2 + y^2 + z^2)}{z^2} \end{pmatrix}$$

$$\mathbf{n}_{uv} = \frac{1}{\sqrt{1+h^2}} \begin{pmatrix} h^2x/rz & h^2y/rz \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{D}_{uv} = \begin{pmatrix} -\frac{4p^2rz}{\sqrt{1+h^2}} & 0 \\ 0 & 0 \end{pmatrix}$$

Cylinder

Angle Parametrization

Given the cylindrical parametrization:

$$[x \ y \ z] = [\cos q \ \sin q \ z]$$

we have the Jacobian matrix:

$$\mathbf{J}_{qf} = \begin{bmatrix} -\sin q & \cos q & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H}_{qf} = \begin{bmatrix} -\cos q & -\sin q & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G}_{qf} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

the normal:

$$\mathbf{n} = [x \ y \ 0]$$

and the second fundamental form:

$$\mathbf{D}_{qf} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Unit Parametrization

With the parametrization:

$$[x \ y \ z] = [r \cos 2p u \ r \sin 2p u \ h v]$$

we have the Jacobian matrix:

$$\mathbf{J}_{uv} = \begin{bmatrix} -2p y & 2p x & 0 \\ 0 & 0 & h \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H}_{uv} = \begin{bmatrix} -4p^2 x & -4p^2 y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G}_{uv} = \begin{bmatrix} 4p^2 r^2 & 0 \\ 0 & h^2 \end{bmatrix}$$

the normal:

$$\mathbf{n} = \frac{1}{r} \begin{pmatrix} -x \\ y \\ 0 \end{pmatrix}$$

and the second fundamental form:

$$\mathbf{D}_{uv} = \frac{1}{r} \begin{pmatrix} -2x & 0 \\ 0 & 0 \end{pmatrix}$$

Torus

Angle Parametrization

Given the torus parametrization:

$$[x \ y \ z] = [(R + r \cos f) \cos q \ (R + r \cos f) \sin q \ r \sin f]$$

we have the Jacobian matrix:

$$\mathbf{J}_{qf} = \begin{bmatrix} \frac{\partial x}{\partial q} & \frac{\partial x}{\partial f} \\ \frac{\partial y}{\partial q} & \frac{\partial y}{\partial f} \\ \frac{\partial z}{\partial q} & \frac{\partial z}{\partial f} \end{bmatrix} = \begin{bmatrix} -(R + r \cos f) \sin q & -r \sin f \cos q \\ (R + r \cos f) \cos q & -r \sin f \sin q \\ r \cos f & r \cos q \sin f \end{bmatrix}$$

the Hessian tensor:

$$\mathbf{H}_{qf} = \begin{bmatrix} \frac{\partial^2 x}{\partial q^2} & \frac{\partial^2 x}{\partial q \partial f} & \frac{\partial^2 x}{\partial f^2} \\ \frac{\partial^2 y}{\partial q^2} & \frac{\partial^2 y}{\partial q \partial f} & \frac{\partial^2 y}{\partial f^2} \\ \frac{\partial^2 z}{\partial q^2} & \frac{\partial^2 z}{\partial q \partial f} & \frac{\partial^2 z}{\partial f^2} \end{bmatrix} = \begin{bmatrix} -(R + r \cos f) \cos q & -r \cos f \sin q & -r \cos q \sin f \\ -(R + r \cos f) \sin q & -r \cos f \cos q & -r \sin q \sin f \\ -r \sin f & -r \cos f \sin q & -r \cos q \cos f \end{bmatrix}$$

the first fundamental form:

$$\mathbf{G}_{qf} = \begin{bmatrix} \frac{\partial x}{\partial q}^2 + \frac{\partial y}{\partial q}^2 + \frac{\partial z}{\partial q}^2 & \frac{\partial x}{\partial q} \frac{\partial x}{\partial f} + \frac{\partial y}{\partial q} \frac{\partial y}{\partial f} + \frac{\partial z}{\partial q} \frac{\partial z}{\partial f} \\ \frac{\partial x}{\partial q} \frac{\partial x}{\partial f} + \frac{\partial y}{\partial q} \frac{\partial y}{\partial f} + \frac{\partial z}{\partial q} \frac{\partial z}{\partial f} & \frac{\partial x}{\partial f}^2 + \frac{\partial y}{\partial f}^2 + \frac{\partial z}{\partial f}^2 \end{bmatrix}$$

the normal:

$$\mathbf{n} = \frac{1}{r} \begin{bmatrix} -x \\ -y \\ z \end{bmatrix}$$

and the second fundamental form:

$$\mathbf{D}_{qf} = \begin{bmatrix} \frac{\partial^2 x}{\partial q^2} \frac{\partial x}{\partial f} + \frac{\partial^2 x}{\partial q \partial f} \frac{\partial x}{\partial q} + \frac{\partial^2 x}{\partial f^2} \frac{\partial x}{\partial f} & \frac{\partial^2 x}{\partial q^2} \frac{\partial y}{\partial f} + \frac{\partial^2 x}{\partial q \partial f} \frac{\partial y}{\partial q} + \frac{\partial^2 x}{\partial f^2} \frac{\partial y}{\partial f} & \frac{\partial^2 x}{\partial q^2} \frac{\partial z}{\partial f} + \frac{\partial^2 x}{\partial q \partial f} \frac{\partial z}{\partial q} + \frac{\partial^2 x}{\partial f^2} \frac{\partial z}{\partial f} \\ \frac{\partial^2 y}{\partial q^2} \frac{\partial x}{\partial f} + \frac{\partial^2 y}{\partial q \partial f} \frac{\partial x}{\partial q} + \frac{\partial^2 y}{\partial f^2} \frac{\partial x}{\partial f} & \frac{\partial^2 y}{\partial q^2} \frac{\partial y}{\partial f} + \frac{\partial^2 y}{\partial q \partial f} \frac{\partial y}{\partial q} + \frac{\partial^2 y}{\partial f^2} \frac{\partial y}{\partial f} & \frac{\partial^2 y}{\partial q^2} \frac{\partial z}{\partial f} + \frac{\partial^2 y}{\partial q \partial f} \frac{\partial z}{\partial q} + \frac{\partial^2 y}{\partial f^2} \frac{\partial z}{\partial f} \\ \frac{\partial^2 z}{\partial q^2} \frac{\partial x}{\partial f} + \frac{\partial^2 z}{\partial q \partial f} \frac{\partial x}{\partial q} + \frac{\partial^2 z}{\partial f^2} \frac{\partial x}{\partial f} & \frac{\partial^2 z}{\partial q^2} \frac{\partial y}{\partial f} + \frac{\partial^2 z}{\partial q \partial f} \frac{\partial y}{\partial q} + \frac{\partial^2 z}{\partial f^2} \frac{\partial y}{\partial f} & \frac{\partial^2 z}{\partial q^2} \frac{\partial z}{\partial f} + \frac{\partial^2 z}{\partial q \partial f} \frac{\partial z}{\partial q} + \frac{\partial^2 z}{\partial f^2} \frac{\partial z}{\partial f} \end{bmatrix}$$

using the torus's implicit equation:

$$(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2$$